Appendix

This technical appendix develops theory to suggest how one can construct indexes of regime shift that amplify the signal of interest and suppress noise that is not of interest. Intuition suggests that the indicator should put as much weight as possible on the component that is approaching bifurcation. Components that are not approaching bifurcation merely add noise to the indicator, and their contribution to the index should be made as small as possible. The dominant eigenvalue used in our computations is one way of achieving the desired weighting. Ecologists are familiar with the use of eigenvalues in community ordination, where the dominant eigenvalue is associated with the principal component of variation in community structure (Pielou 1977). Specifically, if we observe a random multivariate process, the line that explains most of the variance of the multivariate cloud is the eigenvector corresponding to the maximum eigenvalue of the variance-covariance matrix computed from the data (Pielou 1977). This maximum, or dominant, eigenvalue measures the variance explained by this line. We use the dominant eigenvalue in an analogous way, to measure the largest component of variance in system dynamics. In the technical discussion below, we explain why the dominant eigenvalue is a good indicator of potential regime shift in a multivariate ecosystem.

Technical details related to variance near the critical point are developed below. We wish to consider indices for the system

$$(1)\quad dP_i = [M_i + D(P_j - P_i) - c_i P_i - \frac{P_i s m^q}{m^q + P_i^q}]dt + \sigma_i dW_i$$

where i=1,2 and dW_1 is independent of dW_2. Write (1) as follows

$$(2)\quad \begin{aligned} dP_i &= [M_i - c_i P_i - \frac{P_i s m^q}{m^q + P_i^q}]dt + D(P_j - P_i)dt + \sigma_i dW_i \\ dP_i &\equiv \frac{\partial}{\partial P_i} F_i(P_i, M_i)dt + D(P_j - P_i)dt + \sigma_i dW_i \end{aligned}$$

The potential F is given by

$$(3)\quad F = F_1 + F_2 - \frac{D(P_1 - P_2)^2}{2}$$

Let us simplify the analyis by putting $\sigma_1 = \sigma_2 = \sigma$. Put $x=[P_1\ P_2]'$, $a=M_1$, $W=[W_1\ W_2]'$, F(x,a) equal to the right side of (3), and $V = \partial F/\partial x$. In this notation, we may write (1) and (2) as the system,

$$(4)\quad dx = Vdt + \sigma dW$$

Note that E{dW·dW'} = Idt where I is the 2x2 identity matrix. Note that in the applications below we may parametrize an arc through the parameter space (M_1,M_2,D) by the parameter a and still work within a context where a is one dimensional.

It is known (Berglund and Gentz (2002a,b) that the stationary probability density is given by

$$(5)\quad p(x,a) = \exp[bF(x,a)]/N; \quad b \equiv \frac{2}{\sigma^2}$$

where N is a normalization factor. Assume the global maximum, x(a) of F(x,a) is unique and all eigenvalues of the Hessian matrix, $\partial^2 F/\partial x^2$ are negative, for all a < a_c, where a_c is a critical value of a where the largest eigenvalue of $\partial^2 F/\partial x^2$ first passes through zero. Expand F(x,a) in a Taylor series about x(a) to obtain

(6)

$$\begin{aligned}
F(x,a) &= F(x(a),a) + \frac{\partial F(x(a),a)}{\partial x}(x - x(a)) + \frac{1}{2}(x - x(a))'\frac{\partial^2 F(x(a),a)}{\partial x^2}(x - x(a)) + o(|x - x(a)|^2) \\
&= F(x(a),a) + 0 + \frac{1}{2}(x - x(a))'\frac{\partial^2 F(x(a),a)}{\partial x^2}(x - x(a)) + o(|x - x(a)|^2) \\
&= F(x(a),a) + \frac{1}{2}(x - x(a))'\frac{\partial^2 F(x(a),a)}{\partial x^2}(x - x(a)) + o(|x - x(a)|^2)
\end{aligned}$$

where the zero follows from the first order necessary condition for x(a) to be a maximum of F(x,a) and $o(|z|^2)/|z|^2 \to 0$ as $|z| \to 0$, where |z| denotes the Euclidean norm of the vector z.

Insert (6) into (5) and cancel the term F(x(a),a) which is common to both numerator and denominator to obtain

$$(7)\quad p(x,a) = \frac{1}{N}\exp\left\{b\left[\frac{1}{2}(x - x(a))'\frac{\partial^2 F(x(a),a)}{\partial x^2}(x - x(a)) + o(|x - x(a)|^2\right]\right\}$$

where we continue to use N for the normalization factor. Following Berglund and Gentz (2002b), we assume sigma is small enough so the approximation

$$(8)\quad q(x,a) = \frac{1}{N}\exp\left\{b\left[\frac{1}{2}(x - x(a))'\frac{\partial^2 F(x(a),a)}{\partial x^2}(x - x(a))\right]\right\}$$

is good. But (8) is a Normal distribution for y(a) ≡ x-x(a) which has mean vector zero and variance covariance matrix

$$(9)\quad S(a) = -\frac{1}{b}\left(\frac{\partial^2 F(x(a),a)}{\partial x^2}\right)^{-1} = -\frac{\sigma^2}{2}\left(\frac{\partial^2 F(x(a),a)}{\partial x^2}\right)^{-1}$$

We see right away for the scalar case that S(a) goes to infinity as a → a_c. This is so because $\partial^2 F/\partial x^2 \rightarrow 0$ as a → a_c. This observation is the basic foundation for the variance-based indicators which are developed via numerical evidence in this paper. We will sketch the matrix case below.

We believe that a completely rigorous treatment can be developed by adapting Berglund and Gentz (2002a,b) but that task is beyond the scope of the current paper. The basic idea is to move the parameter a slowly enough through time and to assume that σ is small enough so that

(i) The system relaxes close enough to the stationary distribution that we can well approximate the distribution at any date by the stationary distribution for a(t) at date t.

(ii) σ is small enough and F(x,a) is smooth enough so that the potential F(x(a),a) may be approximated by the quadratic

$$(10)\quad F(x(a),a) + (x - x(a))'\frac{\partial^2 F(x(a),a)}{\partial x^2}(x - x(a))$$

where the matrix $\partial^2 F/\partial x^2$ is evaluated at (x(a),a), for each a=a(t). See Berglund and Gentz (2000b) for this type of development.

Returning to the matrix case, we must develop the idea that S(a) becomes "large" as a → a_c and we must develop indices that are dominated by the part of S(a) that contains the most information about impending regime change. For example if S(a) is diagonal, the most information is contained in the largest eigenvalue, which is the smallest eigenvalue of $-\partial^2 F/\partial x^2$. In more detail consider first the case D=0. In this case $\partial^2 F/\partial x^2$ is the diagonal matrix with the second order partial $\partial^2 F/\partial P_i^2$ on the i^{th} diagonal. Thus, letting $y(a) \equiv x - x(a)$ and letting $w \equiv (w1,w2)$ be weights whose squares sum to one, we have $w \cdot y(a)$ is normally distributed with mean zero and variance,

$$(11)\quad \mathrm{var}(w, y(a)) = w_1^2 S_{11} + w_2^2 S_{22} = \frac{\sigma^2}{2}\left[w_1^2\left(\frac{\partial^2 F}{\partial P_1^2}\right)^{-2} + w_2^2\left(\frac{\partial^2 F}{\partial P_2^2}\right)^{-2}\right]$$

and this convex function (in w) takes a maximum at the extreme points of the convex set of w's, i.e. the disk

$$C = \{w \mid w_1^2 + w_2^2 \le 1\}$$

The biggest value is attained by placing all weight on the smallest $|\partial^2F/\partial P_i^2|$. This result is general in the sense that the biggest value is attained by the largest eigenvector on the boundary of the disk C of an appropriate variance-covariance matrix (Goldberger 1991). When D>0, the matrix of cross partial derivatives is no longer diagonal so we have

(12) $E\left[(w'dx)(w'dx)' = w'V\sigma\sigma'V'w\right]$

which is a positive definite quadratic form even if V is not the derivative of a potential and even if the noises dW_1, dW_2 are correlated. The maximum is always taken on the boundary of the disk C at the eigenvector associated with the largest eigenvalue of the positive definite matrix in equation 12.

Finally we point out that there are techniques available to develop useful approximations for systems where there is no potential provided the variance of the driving noise σ^2 is small enough (see Magill (1977) and references therein to the stochastic approximation literature). Indeed Magill (1977) even shows how to develop local spectral analysis for systems that are optimally controlled. This is a much more difficult task than development of approximations for systems under a fixed control as discussed here. We believe that there are fruitful generalizations available for systems that do not have a potential and we believe that the results will end up focusing on the eigenvalue structure of linearization matrices similar to the development of this paper.

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